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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 1

8 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

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Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

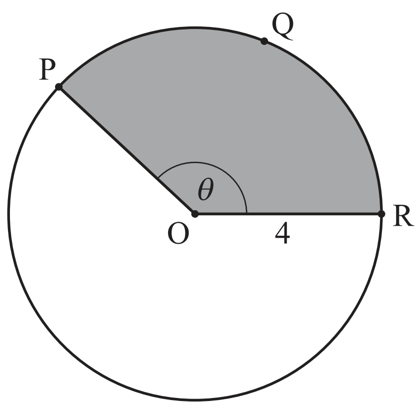
Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows a circle with centre O and radius 4 cm.

diagram not to scale



The points P, Q and R lie on the circumference of the circle and $\text{P}\hat{\text{O}}\text{R} = \theta$, where θ is measured in radians.

The length of arc PQR is 10 cm.

(a) Find the perimeter of the shaded sector. [2]

(b) Find θ. [2]

(c) Find the area of the shaded sector. [2]

(This question continues on the following page)

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(Question 1 continued)

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2. [Maximum mark: 5]

A function f is defined by $f(x) = 1 - \frac{1}{x-2}$, where $x \in \mathbb{R}$, $x \neq 2$.

(a) The graph of $y = f(x)$ has a vertical asymptote and a horizontal asymptote.

Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. [2]

(b) Find the coordinates of the point where the graph of $y = f(x)$ intersects

(i) the y-axis;

(ii) the x-axis. [2]

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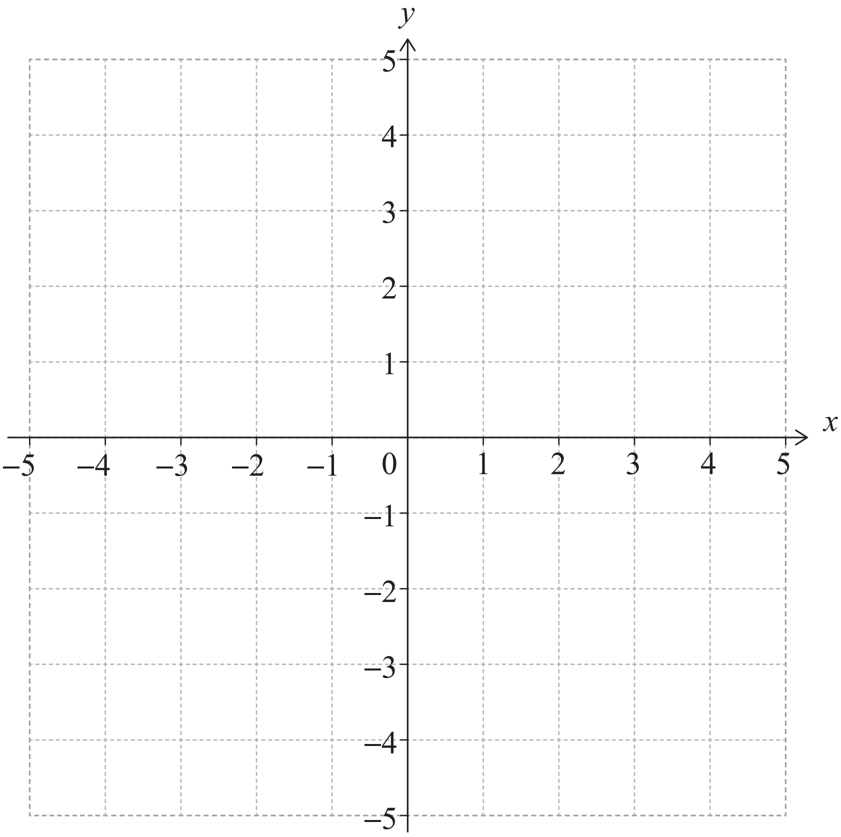
(Question 2 continued)

(c) On the following set of axes, sketch the graph of $y = f(x)$, showing all the features found in parts (a) and (b). [1]



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3. [Maximum mark: 5]

Events A and B are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(A|B) = 0.25$ and $\mathrm{P}(A \cup B) = 0.55$.

Find $\mathrm{P}(B)$.

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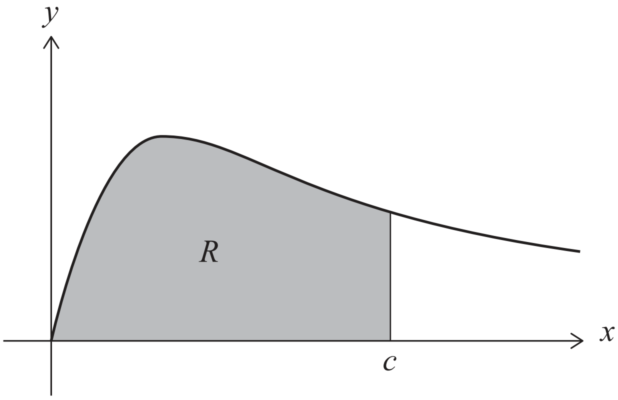
4. [Maximum mark: 6]

The following diagram shows part of the graph of $y = \dfrac{x}{x^2 + 2}$ for $x \geq 0$.



The shaded region R is bounded by the curve, the x-axis and the line $x = c$.

The area of R is $\ln 3$.

Find the value of c.

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5. [Maximum mark: 7]

The functions f and g are defined for $x \in \mathbb{R}$ by

$$f(x) = ax + b\text{, where } a,\ b \in \mathbb{Z}$$

$$g(x) = x^2 + x + 3\,.$$

Find the two possible functions f such that $(g \circ f)(x) = 4x^2 - 14x + 15$.

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6. [Maximum mark: 5]

A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} \frac{1}{2a}, & a \le x \le 3a \\ 0, & \text{otherwise} \end{cases}$$

where a is a positive real number.

(a) State $\mathrm{E}(X)$ in terms of a. [1]

(b) Use integration to find $\mathrm{Var}(X)$ in terms of a. [4]

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7. [Maximum mark: 7]

Use mathematical induction to prove that $\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

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8. [Maximum mark: 7]

The functions f and g are defined by

$$f(x) = \cos x,\ 0 \le x \le \frac{\pi}{2}$$

$$g(x) = \tan x,\ 0 \le x < \frac{\pi}{2}.$$

The curves $y = f(x)$ and $y = g(x)$ intersect at a point P whose x-coordinate is k, where $0 < k < \frac{\pi}{2}$.

(a) Show that $\cos^2 k = \sin k$. [1]

(b) Hence, show that the tangent to the curve $y = f(x)$ at P and the tangent to the curve $y = g(x)$ at P intersect at right angles. [3]

(c) Find the value of $\sin k$. Give your answer in the form $\frac{a+\sqrt{b}}{c}$, where $a,\ c \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$. [3]

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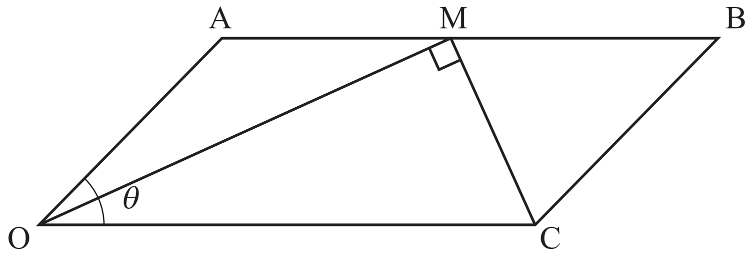
9. [Maximum mark: 9]

The following diagram shows parallelogram OABC with $\overrightarrow{\mathrm{OA}} = \boldsymbol{a}$, $\overrightarrow{\mathrm{OC}} = \boldsymbol{c}$ and $|\boldsymbol{c}| = 2|\boldsymbol{a}|$, where $|\boldsymbol{a}| \neq 0$.



The angle between $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OC}}$ is θ, where $0 < \theta < \pi$.

Point M is on [AB] such that $\overrightarrow{\mathrm{AM}} = k\overrightarrow{\mathrm{AB}}$, where $0 \leq k \leq 1$ and $\overrightarrow{\mathrm{OM}} \bullet \overrightarrow{\mathrm{MC}} = 0$.

(a) Express $\overrightarrow{\mathrm{OM}}$ and $\overrightarrow{\mathrm{MC}}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$. [2]

(b) Hence, use a vector method to show that $|\boldsymbol{a}|^2\ (1 - 2k)\big(2\cos\theta - (1 - 2k)\big) = 0$. [3]

(c) Find the range of values for θ such that there are two possible positions for M. [4]

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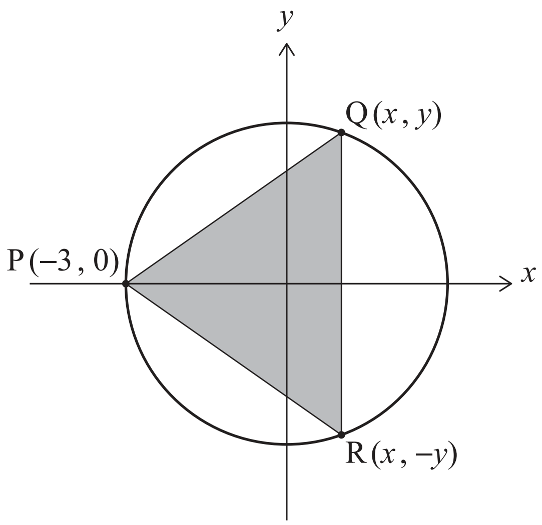
Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 14]

A circle with equation $x^2 + y^2 = 9$ has centre $(0, 0)$ and radius 3.

A triangle, PQR, is inscribed in the circle with its vertices at $\text{P}(-3, 0)$, $\text{Q}(x, y)$ and $\text{R}(x, -y)$, where Q and R are variable points in the first and fourth quadrants respectively. This is shown in the following diagram.



(a) For point Q, show that $y = \sqrt{9 - x^2}$. [1]

(b) Hence, find an expression for A, the area of triangle PQR, in terms of x. [3]

(c) Show that $\frac{\mathrm{d}A}{\mathrm{d}x} = \frac{9 - 3x - 2x^2}{\sqrt{9 - x^2}}$. [4]

(d) Hence or otherwise, find the y-coordinate of R such that A is a maximum. [6]

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Do **not** write solutions on this page.

11. [Maximum mark: 22]

Consider the complex number $u = -1 + \sqrt{3}\,\mathrm{i}$.

(a) By finding the modulus and argument of u, show that $u = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$. [3]

(b) (i) Find the smallest positive integer n such that u^n is a real number.

(ii) Find the value of u^n when n takes the value found in part (b)(i). [5]

(c) Consider the equation $z^3 + 5z^2 + 10z + 12 = 0$, where $z \in \mathbb{C}$.

(i) Given that u is a root of $z^3 + 5z^2 + 10z + 12 = 0$, find the other roots.

(ii) By using a suitable transformation from z to w, or otherwise, find the roots of the equation $1 + 5w + 10w^2 + 12w^3 = 0$, where $w \in \mathbb{C}$. [9]

(d) Consider the equation $z^2 = 2z^*$, where $z \in \mathbb{C}$, $z \neq 0$.

By expressing z in the form $a + b\mathrm{i}$, find the roots of the equation. [5]

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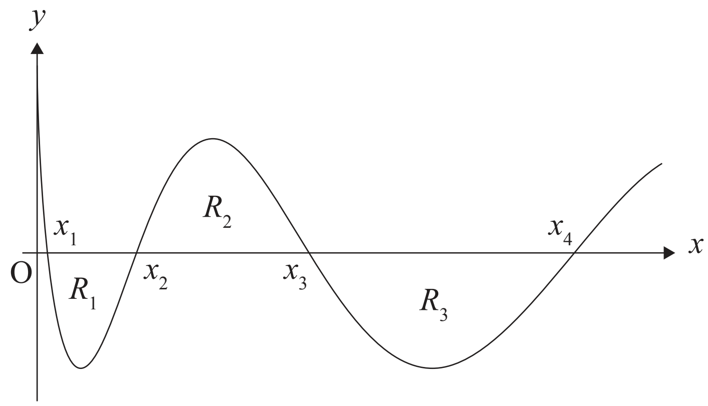
12. [Maximum mark: 17]

(a) By using an appropriate substitution, show that $\int \cos\sqrt{x}\,\mathrm{d}x = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$. [6]

The following diagram shows part of the curve $y = \cos\sqrt{x}$ for $x \geq 0$.



The curve intersects the x-axis at $x_1, x_2, x_3, x_4, \ldots$.

The nth x-intercept of the curve, x_n, is given by $x_n = \frac{(2n-1)^2 \pi^2}{4}$, where $n \in \mathbb{Z}^+$.

(b) Write down a similar expression for x_{n+1}. [1]

The regions bounded by the curve and the x-axis are denoted by $R_1, R_2, R_3, \ldots$, as shown on the above diagram.

(c) Calculate the area of region R_n.

Give your answer in the form $kn\pi$, where $k \in \mathbb{Z}^+$. [7]

(d) Hence, show that the areas of the regions bounded by the curve and the x-axis, $R_1, R_2, R_3, \ldots$, form an arithmetic sequence. [3]

References:

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Answers written on this page
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